# Natural Numbers and their Square Roots expressed by constant Phi and 1


- Harry K. Hahn -

Germany


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## Abstract :

All natural numbers ( 1, 2, 3,....) can be calculated only by using constant Phi (φ) and 1.
I have found a way to express all natural numbers and their square roots with simple algebraic terms, which are only based on Phi (φ) and 1. Further I have found a rule to calculate all natural numbers >10 and their square roots with the help of a general algebraic term. The constant Pi (π) can also be expressed only by using the constant φ and 1 !


## Introduction :

**The asymptotic ratio of successive Fibonacci numbers leads to the golden ratio constant φ ( or Φ )**

Fibonacci Sequences describe morphological patterns in a wide range of living organisms. This is one of the most remarkable organizing principles mathematically describing natural phenomena.

The constant φ is the positive solution of the following quadratic equation :

$$x + 1 = x^2$$

$$\rightarrow \quad \varphi = \frac{1+\sqrt{(5)}}{2} = 1.618034...$$

The **Fibonacci Numbers** defined by **φ** :

1/1 = 1
2/1 = 2
3/2 = 1.5
5/3 = 1.667
8/5 = 1.6
13/8 = 1.625
21/13 = 1.615
34/21 = 1.619
55/34 = 1.618

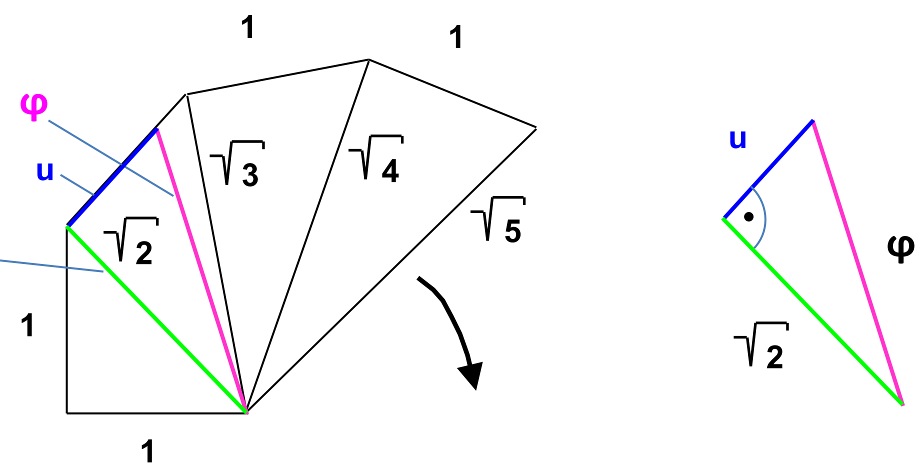
$\downarrow$ **φ**

Because the value of constant φ is close to the square root of **2** and the square root of **3** , I have drawn φ into the start section of the **Square Root Spiral** in order to find a way to calculate the short cathetus **u** of the right triangle φ , square root of **2** and **u** , and to see which relation the cathetus **u** has to the other triangles of the Square Root Spiral :

**The start of the Square Root Spiral is shown with the constant φ drawn in :**

$$\varphi = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+..}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}}$$




The periodic continued fractions of φ and square root of **2** show a very simple structure. But what is with cathetus **u** ?

Now I calculated the numerical value of chatetus **u** with the help of the **Pythagorean Theorem** :

From the right triangle $\varphi$ , square root of **2** & **u** follows :

$$\varphi^2 = (\sqrt{2})^2 + u^2$$ ; application of the Pythagorean Theorem

$$\rightarrow \quad u = \sqrt{\varphi^2 - 2} = 0{,}786151377.....$$ ; we can calculate this value of **u** with the calculator

But because this numerical value doesn't say much, I did some research in the internet with Google, and I actually found an algebraic term which obviously has the same numerical value !

This is the following term :

$$\frac{\sqrt{2\sqrt{5} - 2}}{2} = 0{,}786151377... = u$$

This value is shown in **equation 4.10. on page 11** of the following study : **weblink :** https://arxiv.org/pdf/0706.2043.pdf

The title of the mentioned study :

„PHASE SPACES IN SPECIAL RELATIVITY : TOWARDS ELIMINATING GRAVITATIONAL SINGULARITIES"

by Peter Danenhower

**With the help of the found algebraic term I carried out the following algebraic calculations :**

$$\sqrt{\varphi^2 - 2} = \frac{\sqrt{2\sqrt{5} - 2}}{2}$$ ; I equated the two algebraic terms which obviously represent the same constant !

$$\rightarrow \quad 4\varphi^2 - 8 = 2\sqrt{5} - 2$$ ; I squared both sides and transformed

$$\varphi^2 = \frac{\sqrt{5} + 3}{2}$$ ; ( 1 ) I solved for $\boldsymbol{\varphi^2}$

$$\sqrt{5} = 2\varphi^2 - 3$$ ; ( 2 ) I solved for $\sqrt{5}$

Now I went back to the Square Root Spiral and used the following right triangle :

$$(\sqrt{6})^2 = (\sqrt{5})^2 + 1^2$$ ; application of the Pythagorean theorem

$$6 = (2\varphi^2 - 3)^2 + 1$$ ; I replaced $\sqrt{5}$ by equation ( 2 ) and transformed

$$\rightarrow \quad 3 = \frac{\varphi^4 + 1}{\varphi^2} \quad (3) \quad \rightarrow \quad \sqrt{3} = \sqrt{\frac{\varphi^4 + 1}{\varphi^2}} \quad (4)$$ ; square root 3 expressed by $\varphi$ and 1 !

Now I used the following right triangle :

$$(\sqrt{3})^2 = (\sqrt{2})^2 + 1^2$$ ; application of the Pythagorean theorem and inserting equation ( 3 )

$$\rightarrow \quad 2 = \frac{\varphi^4 + 1}{\varphi^2} - 1 \quad \rightarrow \quad 2 = \frac{\varphi^4 - \varphi^2 + 1}{\varphi^2} \quad (5) \quad \text{and} \quad \sqrt{2} = \sqrt{\frac{\varphi^4 - \varphi^2 + 1}{\varphi^2}} \quad (6)$$

Then I inserted equation ( 3 ) in equation ( 2 ) :

$$\rightarrow \quad \sqrt{5} = 2\varphi^2 - \frac{\varphi^4 + 1}{\varphi^2} \quad \rightarrow \quad \sqrt{5} = \frac{\varphi^4 - 1}{\varphi^2} \;;\; (7.0) \quad \rightarrow \quad 5 = \left(\frac{\varphi^4 - 1}{\varphi^2}\right)^2 \quad (7.1)$$

And I used the following right triangle :

$$(\sqrt{6})^2 = (\sqrt{5})^2 + 1^2$$ ; application of the Pythagorean theorem and inserting equation ( 7.1 )

$$\rightarrow \quad 6 = \left(\frac{\varphi^4 - 1}{\varphi^2}\right)^2 + 1 \quad \rightarrow \quad 6 = \frac{\varphi^8 - \varphi^4 + 1}{\varphi^4} \quad (8) \quad \text{and} \quad \sqrt{6} = \sqrt{\frac{\varphi^8 - \varphi^4 + 1}{\varphi^4}} \quad (9)$$

I continued and used the following right triangles of the **Square Root Spiral** (**SRS**) to calculate the next square roots :

$$(\sqrt{7})^2 = (\sqrt{6})^2 + 1^2$$ ; application of the Pythagorean theorem and inserting equation ( 8 )

$$\rightarrow \quad 7 = \frac{\varphi^8 + 1}{\varphi^4} \quad (10) \quad \rightarrow \quad \sqrt{7} = \sqrt{\frac{\varphi^8 + 1}{\varphi^4}} \quad (11)$$

In the same way I calculated the following square roots and natural numbers with the next right triangles of the **SRS** :

$$\rightarrow \quad 8 = \frac{\varphi^8 + \varphi^4 + 1}{\varphi^4} \quad (12) \quad \text{and} \quad \sqrt{8} = \sqrt{\frac{\varphi^8 + \varphi^4 + 1}{\varphi^4}} \quad (13)$$

$$\rightarrow \quad 10 = \frac{\varphi^8 + 3\varphi^4 + 1}{\varphi^4} \quad (14) \quad \text{and} \quad \sqrt{10} = \sqrt{\frac{\varphi^8 + 3\varphi^4 + 1}{\varphi^4}} \quad (15)$$

$$\rightarrow \quad 11 = \frac{\varphi^8 + 4\varphi^4 + 1}{\varphi^4} \quad (16) \quad \text{and} \quad \sqrt{11} = \sqrt{\frac{\varphi^8 + 4\varphi^4 + 1}{\varphi^4}} \quad (17)$$

$$\rightarrow \quad 12 = \frac{\varphi^8 + 5\varphi^4 + 1}{\varphi^4} \quad (18) \quad \text{and} \quad \sqrt{12} = \sqrt{\frac{\varphi^8 + 5\varphi^4 + 1}{\varphi^4}} \quad (19)$$

**From the above shown formulas** ( equations 12 to 19 ), **I realized a general rule for all Natural Numbers > 10 :**

$$\rightarrow \quad (10+n) = \frac{\varphi^8 + (3+n)\varphi^4 + 1}{\varphi^4} \quad (20) \quad \text{and} \quad \sqrt{(10+n)} = \sqrt{\frac{\varphi^8 + (3+n)\varphi^4 + 1}{\varphi^4}} \quad (30)$$

**For n → ∞**

with $n \in N = \{0, 1, 2, 3, 4, \ldots\}$

**<u>Note</u> :** → The expression (3+n) in the rule can be replaced by products and/or sums, of the equations ( 3 ) to ( 13 ) and number 1 , in order to have final expressions only based on φ and 1 !

**With these general equations (20) and (30) all natural numbers and their square roots can be expressed by only using constant φ and 1 !**

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**The constant Pi (π) can also be expressed by only using the constant φ and 1 ! :**

**I use Viete's formula from the year 1593 :**

$$\pi = \frac{2}{\sqrt{2}} \frac{2}{\sqrt{2+\sqrt{2}}} \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots$$

→ It is also possible to derive from Viète's formula a related formula for π that involves nested square roots of two, but uses only one multiplication :

$$\pi = \lim_{k\to\infty} 2^k \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}}_{k \text{ square roots}}$$

I replace the number **2** in the above shown formulas by the found equation ( 5 ) where number **2** can be expressed by constant **φ** and **1**. Then the constant **Pi (π)** can be expressed by only using the constant **φ** and **1** !

I replaced Number **2** in the above shown formula on the righthand side, with equation ( 5 ) :

$$\pi = \lim_{k\to\infty} \left[\frac{\varphi^4 - \varphi^2 + 1}{\varphi^2}\right]^k \underbrace{\sqrt{\frac{\varphi^4 - \varphi^2 + 1}{\varphi^2} - \sqrt{\frac{\varphi^4 - \varphi^2 + 1}{\varphi^2} + \sqrt{\frac{\varphi^4 - \varphi^2 + 1}{\varphi^2} + \cdots + \sqrt{\frac{\varphi^4 - \varphi^2 + 1}{\varphi^2}}}}}}_{k \text{ square roots}} \quad (40)$$

It seems that the irrationality of Pi (π) is fundamentally based on the constant φ and 1, in the same way as the irrationality of all irrational square roots, and all natural numbers seems to be based on constant φ & 1 !

This is an interesting discovery because it allows to describe many basic geometrical objects like the Platonic Solids only with φ & 1 !

Constant φ and Number 1 ( the base unit ) may represent something like fundamental „space structure constants“ !

## References :


**Phase spaces in Special Relativity : Towards eliminating gravitational singularities -** by Peter Danenhower
see weblink : https://arxiv.org/pdf/0706.2043.pdf


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## further interesting References to the subject :


**The Ordered Distribution of Natural Numbers on the Square Root Spiral** - by Harry K. Hahn
http://front.math.ucdavis.edu/0712.2184 PDF : http://arxiv.org/pdf/0712.2184

**The Distribution of Prime Numbers on the Square Root Spiral** – by Harry K. Hahn
http://front.math.ucdavis.edu/0801.1441 PDF : http://arxiv.org/pdf/0801.1441

**The golden ratio Phi (φ) in Platonic Solids :** http://www.sacred-geometry.es/?q=en/content/phi-sacred-solids

**Number Theory as the Ultimate Physical Theory -** by **I. V. Volovich** / Steklov Mathematical Institute
Study : http://cdsweb.cern.ch/record/179558/files/198708102.pdf

**Letters of Albert Einstein, including his letter to natural constants from 13th October 1945** ( in german language )
http://docplayer.org/69639849-Ilse-rosenthal-schneider-begegnungen-mit-einstein-von-laue-und-planck.html
description of the book contents in english : http://blog.alexander-unzicker.com/?p=27